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EVASION OF A FIXED SPHERE BY A DYNAMICAL OBJECT DRIVEN BY A BOUNDED FORCE†

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A solution is obtained of the problem of synthesizing the control of the motion of a dynamical object (a point mass) evading a fixed spherical obstacle under the action of a bounded force. The set of all points for which evasion is possible is constructed in phase space (of arbitrary dimension), and control modes are constructed for bounded (fixed) and unbounded time intervals. The characteristics of the optimal motion, in particular, the time and minimum distance, are determined for specific initial data. The qualitative properties of the controlled motion are established. © 2003 Elsevier Science Ltd. All rights reserved.

The problems of the control of motion and optimization under conditions of uncertainty and conflict when there are phase constraints have been treated in numerous publications ([1–7] and others). However, in order to construct reference controls and estimates for applications, it is of considerable interest to achieve a complete, accurate solution of problems of the control of motion in model formulations, under simplifying assumptions concerning the structure of the system as well as the nature and form of the constraints. In this paper we investigate the problem of the optimum evasion of a fixed geometrical obstacle of spherical shape by a dynamical object of small linear dimensions (a point mass, driven by a bounded force. The problem of a spherical object evading a sphere or point is equivalent to that considered here.

The formulation of the problem presented here arises in a natural way when constructing time-optimal steering of a dynamical object to a sphere at zero velocity (a "soft landing") [8]. As it turns out, under certain initial conditions the optimum trajectories may intersect the surface once or twice. The question arises of determining the set of phase points (initial data) for which a "soft landing" (or evasion) is possible without intersecting the sphere. In that context one can investigate cases of motion both outside and inside the sphere. In what follows, to fix out ideas, we shall consider the first case (external trajectories) as offering greater interest for flight mechanics. We note that the problem of time-optimal intersection of a sphere [9] is in a sense the "inverse" of the problem considered below, namely, the evasion of a spherical domain under the action of a force of bounded magnitude.

1. FORMULATION OF THE PROBLEM

Let us consider the motion of a point mass m in Euclidean n-space ($n \geqslant 1$), driven by a control force F of bounded magnitude; we will assume the presence of an impenetrable sphere (obstacle) with arbitrary fixed centre x_0 and radius $R > 0$ [8]

$$\begin{gathered} m\ddot{x} = F, \quad |F| \le F_0, \quad x(0) = x^0, \quad \dot{x}(0) = \upsilon^0 \\ S_R^n = \{x: |x - x_0| \le R\}, \quad x^0 \notin S_R^n \quad (|x^0 - x_0| > R) \end{gathered} \tag{1.1}$$

We will consider for the control system (1.1) the problem of the geometrical point x evading the obstacle S_R^n at any instant of time $t > 0$, that is, $|x(t) - x_0| > R$, no restrictions being imposed on the velocity $\dot{x} = \upsilon$. Naturally, the problem is not solvable for arbitrary initial data x^0 and υ^0. Moreover, it is meaningful if the velocity vector υ^0 is inside a cone with vertex at the point $x = x^0$ and generators touching the sphere.

System (1.1) is defined by a set of $3(n + 1)$ parameters: the scalars m, F_0, R and the vectors x^0, υ^0, x_0. We define dimension less variable $u = F/F_0$, $(x - x_0)/R \to x$, $t/\tau \to t$, $\upsilon\tau/R \to \upsilon$, $\tau = (mR/F_0)^{1/2}$. Then, instead of system (1.1), we obtain [8]

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$$\dot{x} = v, \quad \dot{v} = u, \quad x(0) = x^0, \quad v(0) = v^0; \quad |u| \le 1$$
$$S_1^n = \{x: |x| \le 1\}, \quad |x^0| > 1 \tag{1.2}$$

By central symmetry, the general problem for $n \geq 2$ is equivalent to the problem in a plane ($n = 2$). If the vectors x^0 and v^0 are non-collinear, this plane is uniquely defined; otherwise, it is determined apart from an arbitrary rotation. The case $n = 1$ (motion along a straight line, with S_1^1 a closed interval) is fairly simple and of no particular interest.

Let us consider the attainable set $D_0(t, x^0, v^0)$ for system (1.2) (see, e.g. [3]). Since there are no restrictions on the velocity of the point, we shall study this set only in coordinate space. The evasion problem has a solution if in D at least one trajectory exists, each point of which is at unit distance (or greater) from the origin. We know that in the case under consideration the attainable set is compact and convex. Then we can study only its boundary points. To find these points, we must solve an optimal control problem with objective functional

$$(c_x, x(T)) \to \min_u, \quad |u| \le 1 \tag{1.3}$$

and system (1.2), where T is an arbitrary fixed instant of time and c_x is an arbitrary unit vector. Having considered all possible vectors c_x in (1.3), we can construct the required boundary of the attainable set at time T. For that instant of time, we must find, among all points of the boundary, that farthest from the origin

$$J = |x(T)| \to \max_{c_x}, \quad |c_x| \le 1 \tag{1.4}$$

Having solved problem (1.4) for all $T > 0$, we obtain the family of trajectories $x^*(T)$ deviating the farthest from the origin among all possible trajectories of the system. Among these we find the one whose terminal point is closest to the origin:

$$\Phi = |x^*(T)| \to \inf_T, \quad 0 < T < \infty \tag{1.5}$$

If this point is not inside the sphere, then, as will be shown below, the original problem has a solution, and this solution has been found.

We can then proceed to study the problem for any initial data and determine the boundary G of the set M of admissible initial data, where

$$M = \{x, v: \Phi(x, v) > 1\}, \quad G = \{x, y: \Phi(x, v) = 1\} \tag{1.6}$$

Thus, the proposed procedure of optimal evasion involves the successive solution of four problems: (1) determine the boundary points of the attainable set, (2) determine the point of that set farthest from the origin for every time $t = T < \infty$, (3) find the point of the trajectory obtained that is closest to the origin, and (4) determine the boundary of the attainable set of initial data.

The solution of the first two problems is comparatively easy to obtain by analytical methods; it is presented in Section 3. Determination of a minimizing trajectory $x^*(T)$ involves certain computational resources; the solution is presented in Section 3. The determination of the boundary of the attainable set (the fourth problem) turns out to be possible by analytical means; it is given in Section 5.

2. CONSTRUCTION OF THE TRAJECTORY WITH MAXIMUM DEVIATION FROM THE ORIGIN

The desired solution of problem (1.2), (1.3) is constructed using the necessary optimality conditions of the maximum principle [1]; we have

$$\dot{x} = v, \quad \dot{v} = u^*, \quad x(0) = x^0, \quad v(0) = v^0, \quad u^* = q|q|^{-1}$$
$$\dot{p} = 0, \quad \dot{q} = -p, \quad p(T) = -c_x, \quad q(T) = 0 \tag{2.1}$$

where p and q are the variable conjugate to x and v.

It follows from relations (2.1) that p is a constant unit vector. Determination of $q(t)$ and substitution into the expression for u^* yields the expressions

$$u^* = -c_x, \quad v = v^0 - c_x t, \quad x = x^0 + v^0 t - c_x t^2/2, \quad |c_x| = 1 \tag{2.2}$$

To solve problem (1.4) taking (2.2) into consideration, we must find the maximum

$$\begin{aligned} &J[c_x] = \left|x^0 + v^0 T - c_x \frac{T^2}{2}\right| = \left(x_0^{T2} - (x_0^T, c_x)T^2 + \frac{T^4}{4}\right)^{1/2} \to \max_{c_x}, \quad |c_x| \le 1 \\ &J[c_x^*] = |x_0^T| + \frac{T^2}{2}, \quad |x_0^T| = (l^2 + 2clhT + h^2T^2)^{1/2}, \quad c_x^* = -\frac{x_0^T}{|x_0^T|} \\ &x^0 + v^0 T \equiv x_0^T \neq 0, \quad l = |x^0|, \quad h = |v^0|, \quad c = (x^0, v^0) l^{-1} h^{-1} = \cos\alpha \end{aligned} \tag{2.3}$$

Thus, if $|x_0^T| \neq 0$ the solution of problems (1.2), (1.3) and (1.2), (1.4) can be constructed uniquely using relations (2.2) and (2.3). Obviously, if $c \neq -1$ then $|x_0^T| \neq 0$. Let us first consider this general situation; the case $c = -1$ will be investigated separately in Section 4. The optimal control u^* of (2.2), (2.3) has a simple geometrical interpretation: it must be collinear with the non-zero vector x at $t = T$ for $u = 0$.

3. DETERMINATION OF THE POINT CLOSEST TO THE ORIGIN ON THE TRAJECTORY WITH MAXIMUM DEVIATION FROM THE ORIGIN

To solve problem (1.2), (1.5), we consider the functional $J^* = J[c_x^*]$ in (2.3) as a function of the argument $T > 0$ and the parameters $l > 1$, $h > 0$, $-1 < c < 0$. The problem is to determine the minimum $\Phi = \inf_T J^*$ and the corresponding value of T^*. Due to the structural properties of J^*, the desired values of Φ and T^* are functions of the aforementioned three parameters l, h and c. The number of parameters may be reduced by normalizing the functional J^* relative to l and the arguments h and T relative to $\sqrt{l}$, that is, instead of problem (1.5), we consider the minimization problem

$$\begin{aligned} &J^*/l = I(\theta, \gamma, c) \equiv L + \theta^2/2 \to \inf_\theta, \quad L \equiv (1 + 2c\gamma\theta + \gamma^2\theta^2)^{1/2} = |x_0^T| l^{-1} \\ &-1 < c < 0, \quad \gamma = h/\sqrt{l} > 0, \quad \theta = T/\sqrt{l}, \quad 0 < \theta < \infty \end{aligned} \tag{3.1}$$

The first derivative of the function $l(\theta, \gamma, c)$ with respect to θ

$$dI/d\theta = \gamma(c + \gamma\theta)/L(\theta, \gamma, c) + \theta \tag{3.2}$$

considered for any fixed values of $\gamma > 0$ and $-1 < c < 0$, is negative at $\theta = 0$ and positive as $\theta \to \infty$. Let us find the second derivative

$$d^2I/d\theta^2 = \gamma^2(1 - c^2)/L^3 + 1 \tag{3.3}$$

The function (3.3) is positive for any $\theta > 0$, $\gamma > 0$ and $|c| < 1$. Consequently, $l(\theta, \gamma, c)$ always has a unique minimum as a function of θ for the parameter values of interest here. It may be found by equating (3.2) to zero. We have

$$\gamma(c + \gamma\theta) + \theta L(\theta, \gamma, c) = 0, \quad -1 < c < 0, \quad \gamma > 0 \tag{3.4}$$

This relation may be reduced to an equation of the fourth degree in θ by moving θL to the right and squaring both sides of the equation. We obtain

$$\gamma^2\theta^4 + 2c\gamma\theta^3 + (1 - \gamma^4)\theta^2 - 2c\gamma^3\theta - c^2\gamma^2 = 0 \tag{3.5}$$

We can use Cardapo's formulae, and then the only suitable root $\theta^* > 0$ is most easily found by direct substitution into Eq. (3.4).

The corresponding controls and trajectories, according to relations (2.2), (2.3), are

$$u^* = (x^0 + v^0T^*)|x^0 + v^0T^*|^{-1}, \quad T^* = \theta^*|x^0|^{1/2}$$
$$x^*(t) = x^0 + v^0t + u^*t^2/2, \quad v^*(t) = v^0 + u^*t, \quad t > 0 \tag{3.6}$$

We must now verify that the trajectory $x^*(t)$ just found does not have points closer to the origin than $x_*^* = x^*(T^*)$. We define a new independent variable $\delta = t - T^*$. Then

$$x^{*2}(\delta) = (x_*^* + v_*^*\delta + u^*\delta^2/2)^2 = x_*^{*2} + 2(x_*^*, v_*^*)\delta +$$
$$+ (x_*^*, u^*)\delta^2 + v_*^{*2}\delta^2 + (v_*^*, u^*)\delta^3 + u^{*2}\delta^4/4, \quad v_*^* = v^*(T^*) \tag{3.7}$$

The derivative $dx^{*2}(\delta)/d\delta$ vanishes at $\delta = 0$ for the same parameter values as the expression (3.2). Then the scalar product (x_*^*, v_*^*) vanishes. We can infer from (1.3), (1.4) and (2.3) that $u^* = x_*^*/|x_*^*|$. Then the scalar product (v_*^*, u^*) also vanishes. We have

$$x^{*2}(\delta) = x_*^{*2} + (x_*^*, u^*)\delta^2 + v_*^{*2}\delta^2 + u^{*2}\delta^4/4$$
$$d^2x^{*2}/d\delta^2 = 2(x_*^*, u^*) + 2v_*^{*2} + 3u^{*2}\delta^2 \tag{3.8}$$

The second derivative, defined by the second equality of (3.8), is always positive. Consequently, when $t < T^*$ the object approaches the origin at a decreasing approach velocity, and then recedes at a velocity with an increasing radial component. Since $x^{*2}(\delta)$ is symmetric about zero as a function of δ, it follows that the distance varies on approaching the origin according to the same law as when receding.

Let us investigate the optimum value of θ, defined in accordance with Eq. (3.4) as a function of γ and c. Finding the implicit derivative of this function with respect to γ for constant c and equating it to zero, we obtain

$$c + 2\gamma\theta + \theta(c\theta + \gamma\theta^2)/L = 0 \tag{3.9}$$

Equations (3.4) and (3.9) define a curve $\gamma(c)$ (or $c(\gamma)$) for which the function $\theta^*(\gamma, c)$, considered as a function of $\gamma > 0$ for given $c < 0$, attains an extremum. We now transform these relations for constructing the curve, after eliminating the unknown θ. Expressing L using Eq. (3.4) and substituting the result into Eq. (3.9), we obtain the following auxiliary equation

$$\frac{\gamma(c + \gamma\theta)}{\theta} = \theta\frac{c\theta + \gamma\theta^2}{c + 2\gamma\theta} \tag{3.10}$$

Equation (3.10) may be rewritten as an equation with a polynomial in powers of θ on one side and zero on the other. Multiplying this equation by γ and adding the result to Eq. (3.5), we obtain

$$\theta(c\gamma\theta^2 + (1 + \gamma^4)\theta + c\gamma) = 0 \tag{3.11}$$

Since $\theta \neq 0$, this is a quadratic equation in θ.

On the other hand, both sides of Eq. (3.10) may be divided by $c + \gamma\theta$, since Eq. (3.4), considered in the set $c = -\gamma\theta$, has a solution only at $L = 0$ which, according to (2.3) and (3.1), is possible only if $c = -1$. Taking this into consideration, we reduce Eq. (3.10) to the form

$$\theta^3 = 2\gamma^2\theta + c\gamma \tag{3.12}$$

Substituting θ^3 from (3.12) into Eq. (3.5) we obtain a relation which may be used to eliminate terms of degree 0 and 1 in θ. In fast, we have

$$\gamma^2\theta^4 + (1 - \gamma^4)\theta^2 + 2c\gamma^3\theta + c^2\gamma^2 = 0 \tag{3.13}$$

Now, adding (3.13) and (3.5), we obtain an auxiliary equation analogous to (3.11)

$$\gamma^2\theta^2 + c\gamma\theta + 1 - \gamma^4 = 0 \tag{3.14}$$

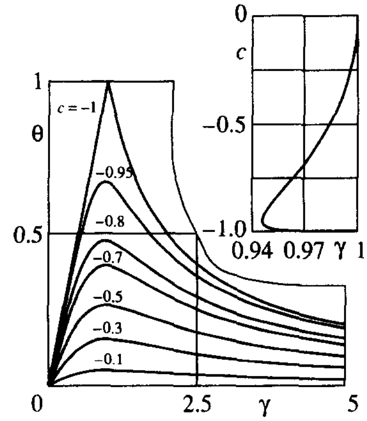

Fig. 1

We now find an expression for θ^2 from (3.14) and substitute the result into Eq. (3.11) divided by θ. The result is a linear equation in θ, from which we find

$$\theta = \frac{c}{\gamma}\frac{1-2\gamma^4}{1+\gamma^4-c^2}, \quad \gamma = \gamma(c) \quad (c = c(\gamma)) \tag{3.15}$$

Now, as follows from (3.4), the function $\theta = \theta(\gamma, c)$ vanishes for constant c at $\gamma = 0$, and $\theta \to 0$ as $\gamma \to +\infty$; hence its unique extremum as a function of γ, as given by (3.15), is a maximum.

We now derive a relation between c and γ which is satisfied for this maximum. Substituting (3.15) into (3.14) we obtain the desired relation between γ and c

$$\frac{c^2\gamma^4(4\gamma^4+c^2-5)-(\gamma^4-1)(\gamma^4+1)^2}{(\gamma^4+1-c^2)^2} = 0 \tag{3.16}$$

The numerator of the fraction (3.16) is a quadratic polynomial in c^2. Its roots

$$c_{1,2}^2 = (2\gamma^2)^{-1}(\gamma^2(5-4\gamma^4) \pm |2\gamma^4-1|(5\gamma^4-4)^{1/2}) \tag{3.17}$$

determine the desired function $c(\gamma)$. Since both roots exist if $\gamma^4 \geqslant 4/5$, it follows that $2\gamma^4 - 1 > 0$. Thus

$$c_{1,2} = -2^{-1/2}\gamma^{-1}(\gamma^2(5-4\gamma^4) \pm (2\gamma^4-1)(5\gamma^4-4)^{1/2})^{1/2} \tag{3.18}$$

It follows from (3.16) that as $c \to 0$ and as $c \to -1$ we have $\gamma \to 1$. Both functions (3.18) are shown in the inset of Fig. 1. At $\gamma = (4/5)^{1/4} \approx 0.946$ (the abscissa of the leftmost point of the curve) the graph of c_1 become the graph of c_2. Interestingly, as c varies from -0 to $-1+0$ the quantity γ varies only slightly (by approximately 6%).

4. THE CASE $c = -1$ AND THE SINGULAR CASE

Let us find the minimum in (3.1) when $c = -1$. We have

$$I(\theta, \gamma, -1) \equiv |1-\gamma\theta| + \theta^2/2 \to \inf_{\theta} \tag{4.1}$$

Then the required value of θ will be

$$\theta^* = \gamma, \quad 0 < \gamma < 1 \tag{4.2}$$

The case in question corresponds to motion along a straight line with a control of absolute value 1 directed against the velocity. It seems natural to call such a control "deceleration". This is the regime obtained in the limit as $c \to -1$ for $0 < \gamma < \gamma^*$, where γ^* is a root of Eq. (3.16). Consequently, we can call this situation "decelerating control" when $c \neq -1$ as well.

For other values of γ we obtain

$$\theta^* = 1/\gamma, \quad \gamma \geq 1 \tag{4.3}$$

This case, as follows from (2.3) and (3.1), corresponds to the singular situation $L = 0$ (this is, $|x_0^T| = 0$). It occurs only when $c = -1$ and $l = hT$. In that case the attainable set at time T is a sphere of radius $T^2/2$ about the origin. Consequently, the extremum in (1.4) is attained for any vector c_x. Nevertheless, the solution (4.3) of problem (1.5) is physically meaningful: it is the time needed to reach the point closest to the origin. Since this point must be on the boundary of the attainable set, the required control u^- is independent of the time.

To determine u^-, we choose any of the planes in the space where the point is moving in such a way that it contains the initial velocity vector. Without loss of generality, we may assume that the initial velocity vector points along the abscissa axis to the origin. The formula for the radius vector of the point is

$$\begin{aligned} &x(t) = x^0 + v^0 t + u^- t^2/2, \quad x^0 = (l, 0), \quad v^0 = (-h, 0) \\ &u^- = (u_1^-, u_2^-), \quad u^- = \text{const}, \quad |u^-| = 1 \end{aligned} \tag{4.4}$$

By (4.4), the square of the radius vector will be

$$x^2(t) = (l - ht)^2 + t^2 u_1^- l - t^3 u_1^- h + t^4/4 \tag{4.5}$$

The point closest to the origin on the trajectory may be determined by equating the derivative $dx^2(t)/dt$ to zero. We obtain

$$-2h(l - hT) + Tu_1^-(2l - 3hT) + T^3 = 0 \tag{4.6}$$

where $T = T(u_1^-)$ is the time needed to reach the point nearest the origin. The desired control u_1^- must maximize the distance from that point to the origin. Consequently, we have to equate the derivative $dx^2(T(u_1^-))/du_1^-$ to zero. Again using (4.5), we obtain

$$\frac{dx^2(T(u_1^-))}{du_1^-} = \frac{\partial x^2}{\partial T}\frac{\partial T}{\partial u_1^-} + \frac{\partial x^2}{\partial u_1^-} \equiv \frac{\partial x^2}{\partial u_1^-} = T^2 l - T^3 h = 0 \tag{4.7}$$

where we have used the fact that the derivative $\partial x^2/\partial T$ vanishes by (4.6). It follows from the last equality of (4.7) that $T = l/h$, as expected. Substituting T into (4.6) we obtain

$$u_1^- = l/h^2, \quad u_2^- = (1 - l^2/h^4)^{1/2} \tag{4.8}$$

Since we are considering the case $\gamma \equiv h/l^{1/2} > 1$, formulae (4.8) are always meaningful and define an optimal control in the singular case (4.3).

As the motion proceeds, the direction of the velocity vector will increasingly deviate from its initial value. Consequently, we may speak of "evasion" when $\gamma > \gamma^*$. But if $\gamma = \gamma^*$, "evasion" smoothly passes into "deceleration" and vice versa. In particular, when $\gamma^* = 1$ formulae (4.8) yield a control directed against the velocity, so that even when $c = -1$ there is no sudden qualitative change.

Now, when all cases of practical interest have been investigated, we can construct the function $\theta = \theta(\gamma)$ for different values of c. It is shown in Fig. 1. All the maxima are very close to 1 on the abscissa axis, but slightly to the left of it, as shown in the inset. The exception is the function obtained for $c = -1$, whose maximum is attained exactly at 1, where the curve has a cusp. The parts of the graphs to the left of the maximum correspond to "decelerating" control, while those to the right correspond to "evading" control.

For every value of θ, there are two γ values, the smaller of which corresponds to the first control mode and the large to the second. This picture is readily explained from the standpoint of mechanics. Suppose the velocity vector at the start of the motion is directed almost to the origin. Then the time required to reach the point closest to this initial point will be the same both for a relatively low starting velocity, at which the object can be quickly stopped, and for a high velocity, when the point rapidly approaches the origin and evasion is achieved. At intermediate velocities, more time is needed either to decelerate or to reach the origin in evasion. Note that, the closer c is to its limiting value $c = -1$, the greater is this difference in time.

5. DETERMINATION OF THE SET OF INITIAL POINTS OF THE PHASE SPACE FOR WHICH EVASION IS POSSIBLE

Let us construct the boundary of the set of vectors x^0 and v^0 for which the object can evade a spherical obstacle at the origin. As already established, the solution of the problem may be expressed in terms of three variables l, γ and c. This property, as observed, indicates the equivalence of the case $n \geqslant 2$ to the case $n = 1$ in the situation of general position.

By (1.6) and (3.1), the admissible set M and its boundary G may be represented by the expressions

$$
\begin{aligned}
&I(\theta^*, \gamma, c) > l^{-1}, \quad ((x, v) \in M)\\
&I(\theta^*, \gamma, c) = l^{-1}, \quad ((x, v) \in G)\\
&x^0 = x \notin S_1^n, \quad v^0 = v, \quad \gamma > 0, \quad -1 \le c < 0, \quad l > 1
\end{aligned}
\tag{5.1}
$$

Consequently, the limiting distance that makes evasion possible is given (implicitly) by the formulae

$$
l(\gamma, c) = (L + \theta^2/2)^{-1}, \quad \gamma(c + \gamma\theta) + \theta L = 0 \tag{5.2}
$$

Since the normalized time θ^* of closest approach is uniquely defined as a function of the parameters γ and c (see Sections 3 and 5), it follows that the function $l(\gamma, c)$ is uniquely constructed. As a result, formulae (5.2) define the boundary G (5.1) of the set of phase points (x, v) for which the control (3.6), (4.8) leads to evasion. The boundary G may be constructed in the three-dimensional space of self-similar variables l, γ and c.

We now construct an explicit relation between l, γ and c. To that end, we solve the second equation in (5.2) for L and substitute the result into the first. We obtain

$$
\theta^3/2 - (\zeta + \gamma^2)\theta - \gamma c = 0, \quad \zeta = l^{-1} \tag{5.3}
$$

The introduction of the parameter ζ is convenient because $0 < \zeta < 1$ for $1 < l < \infty$. Continuing, we square both sides of the first relation in (5.2) and substitute θ^3 from (5.3) into the resulting formula. After some lengthy reduction, we obtain a quadratic relation in θ

$$
(\zeta + \gamma^2)\theta^2 + 3c\gamma\theta + 2(1 - \zeta^2) = 0 \tag{5.4}
$$

Another quadratic equation for θ is obtained if we express θ^2 from (5.4) and substitute the result into (5.3) multiplied by θ. This gives

$$
((3/2)c\gamma\theta^2 + (1 - \zeta^2)\theta)/(\zeta + \gamma^2) + (\zeta + \gamma^2)\theta + \gamma c = 0 \tag{5.5}
$$

Using the auxiliary relations (5.4) and (5.5), we can eliminate θ^2 and obtain a single-valued expression for θ as a function of the given parameters ζ, γ and c. Indeed, substituting the value of θ^2 from (5.4) into (5.5) and solving the resulting linear equation (in θ) for θ, we obtain the following convenient representation

$$
\theta = \frac{-2\gamma c((\zeta + \gamma^2)^2 - 3(1 - \zeta^2))}{2(\zeta + \gamma^2)((\zeta + \gamma^2)^2 + 1 - \zeta^2) - 9\gamma^2 c^2} \tag{5.6}
$$

The desired relation defining the boundary G (5.1) is obtained by substituting expression (5.6) into (5.4). As a result we have the equation of the boundary in the space of the self-similar variables ζ, γ and c.

$$
\begin{aligned}
&27\gamma^4 c^4 - 4\gamma^2 c^2(\zeta + \gamma^2)(9(1 - \zeta^2) + (\zeta + \gamma^2)^2) + 4(1 - \zeta^2)((\zeta + \gamma^2)^2 + 1 - \zeta^2)^2 = 0\\
&0 < \zeta \le 1, \quad \gamma \ge 0, \quad -1 \le c \le 0
\end{aligned}
\tag{5.7}
$$

This is an equation of degree 6 in the unknown $\zeta = \zeta(\gamma^2, c^2)$, or an equation of degree 4 in $\gamma^2 = \gamma^2(\zeta, c^2)$, or a biquadratic equation in c: $c^2 = c^2(\zeta, \gamma^2)$. It is convenient for analytical and numerical investigation, since the self-similar variables ζ and c vary within bounded limits.

However, these variables are not geometrically intuitive for describing the dynamics and trajectories of the controlled motion. It is preferable to express the equation of the boundary G (5.1) in terms of the mechanical variables $l = |x|$, $h = |\upsilon|$, $c = \cos(x, \upsilon)$. So we will express Eq. (5.7) in terms of the initial values l, h and c. Elementary reduction yields the equation of the boundary G (5.1) in the form

$$\begin{gathered} 27h^4l^4c^4 - 4h^2l^2(h^2+1)((h^2+1)^2+9(l^2-1))c^2+4(l^2-1)((h^2+1)^2+l^2-1)^2 = 0 \\ 1 \le l < \infty, \quad h \ge 0, \quad -1 \le c \le 0 \end{gathered} \tag{5.8}$$

This relation may be understood as a biquadratic equation in c, as a cubic equation in l^2, or as an equation of degree 4 in h^2. The necessary roots are most easily determined by substituting into (5.2).

Naturally, it seems most attractive to solve Eq. (5.8) for the unknown c; this yields the expressions

$$\begin{gathered} c_{1,2} = -2^{1/2}(3lh3^{1/2})^{-1}((h^2+1)((h^2+1)^2+9(l^2-1)) \pm ((h^2+1)^2-3(l^2-1))^{3/2})^{1/2} \\ h \ge h_*, \quad h_* = (3(l^2-1))^{1/2}-1 \end{gathered} \tag{5.9}$$

We must now isolate the desired function $c = c(h, l)$ from (5.9). Note that both branches of (5.9), considered for constant l, begin at the same point with $h = h_*$ and have no other common points; moreover, c_1 always lies beneath c_2. In addition. $\lim c_1 = -\infty$ and $\lim c_2 = -(l^2-1)^{1/2}/l$ as $h \to +\infty$. It is clear from general considerations that the graph of the function $c = c(h)$ with l fixed must begin at a point with ordinate -1. As follows from (4.2) and (4.3), the corresponding abscissa is defined differently for different γ values (and consequently for different l values). If $c = -1$, it follows from the second equation of (5.2) for $0 < \gamma < 1$ that $h' = (2(l-1))^{1/2}$, while for $\gamma > 1$ we obtain $h'' = l/2^{1/2}$; in both cases, the corresponding points belong to the branch c_2. Since $\gamma = h/l^{1/2}$, it follows that in the former case $1 < l < 2$ and in the latter $l > 2$. Note that $h_* < h' < h''$ for all $l < 1$ except $l = 2$, when all three values are identical.

Thus, if $1 < l < 2$, the branch c_1 has a unique point with ordinate $c = -1$, namely, with $h = h''$. this point does not satisfy the conditions described above. For all other $h \geqslant h_*$ we have $c_1 < -1$. In the case $l > 2$, if $h > h''$, we also obtain $c_1 < -1$. The final expressions explicitly defining the boundary G (5.1) in the form $c = c(l, h)$ are

$$\begin{gathered} c = -2^{1/2}(3lh3^{1/2})^{-1}((h^2+1)((h^2+1)^2+9(l^2-1)) - ((h^2+1)^2-3(l^2-1))^{3/2})^{1/2} \\ h \ge (2(l-1))^{1/2}, \quad 1 < l < 2 \\ h \ge l/2^{1/2}, \quad l > 2 \end{gathered} \tag{5.10}$$

The family of functions $c = c(h)$ for different constant l values is shown in Fig. 2. For each l, the domain M (5.1) lies to the left of and above the appropriate curve. Note that when $l > 2$ all the graphs are tangent to the abscissa axis, but that is not the case for $l < 2$. In mechanical terms this means that, if at the beginning of the motion the velocity vector points almost exactly to the origin, then, if the initial difference is significant, even a small difference in the angle proves to be decisive in determining whether the obstacle can be evaded. At a small starting distance from the sphere, the value of the angle plays practically no role. If $h \to \infty$, then each of the curves tends to their respective horizontal asymptotes that correspond to $-(l^2-1)^{1/2}/l$. This correspond to the cosine of the angle between the tangent to the unit sphere about the origin from the initial point of the trajectory and the abscissa axis. At smaller angles, obviously, evasion is possible at any initial velocity and the problem formulated in this paper is meaningful only when the velocity vector at the starting time points into the interior of the corresponding come. As can be seen from Fig. 2, the transition from $l < 2$ to $l > 2$ is continuous, without jumps. The point at which the curve for $l = 2$ cuts the abscissa axis corresponds to $h = 2^{1/2}$.

We will also consider the family of function $l = l(h)$ for different constants c. It is illustrated in Fig. 3, For all values of c, the domain M (5.1) lies to the left of and above the appropriate curve. If $h \ll 1$, the function $l(h, c)$ admits of an approximate representation $l \approx 1 + h^2(1 - c^2/2)$. Each of the graphs, except that corresponding to $c = -1$, tends to their respective asymptotes $l = (1 - c^2)^{-1/2}$, for

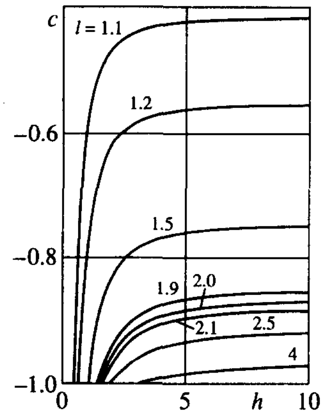

Fig. 2

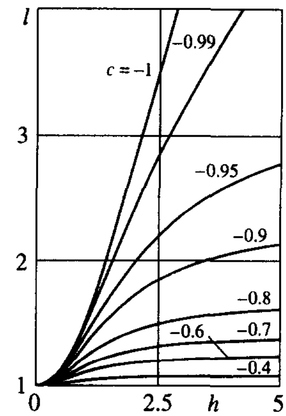

Fig. 3

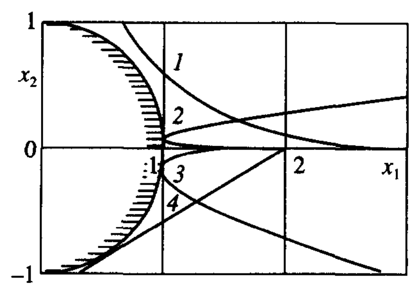

Fig. 4

the reasons enumerated previously. The curve for $c = -1$ is described for $0 < h < 2^{1/2}$ by the formula $l = 1 + h^2/2$; for $h > 2^{1/2}$ it is the straight line $l = 2^{1/2}h$.

To conclude, we will present a few examples of trajectories for different initial data. Figure 4 shows the (x_1, x_2) plane, which at the starting time contains the velocity vector and radius vector of the point mass. Half of the unit sphere which has to be evaded is hatched. All trajectories begin on the abscissa axis (x_1). Curve 1 corresponds to $\alpha = 179.99°, x_1^0 = 3, |\upsilon_0| = 2$. Optimal control of motion with these parameter values enables the point mass to pursue an "evasion mode" to be kept comparatively far from the obstacle. Note that if we had applied a control directed against the velocity ("deceleration"), the point mass could be stopped only on the surface of the sphere itself, and an arbitrarily small increase in initial velocity would have caused collision. Trajectory 2 begins at $\alpha = 179.99°, x_1^0 = 2, |\upsilon_0| = 2^{1/2}$. In that case the control performs in a "deceleration mode", as is clearly seen from the shape of the curve, which almost touches the sphere. If the velocity had been directed exactly towards the origin, the point mass would have moved precisely along the abscissa axis until reaching a complete stop on the sphere surface, when it would have begun to move in the reverse direction. The parameters chosen for trajectory 3 were $\alpha = -175°, x_1^0 = 1.5, |\upsilon^0| \approx 1.0114$, which lead to tangency with the sphere. This may serve as an example of the application of the "evasion mode". Trajectory 4 begins at the point $x_1^0 = 2$, at a high velocity $|\upsilon_0| = 10$, with $\alpha = -150°$. As can be seen from the graph, the control will only slightly deflect this curve from a straight line tangent to the obstacle. Obviously, for such values of x_1^0 and α, evasion is achievable at initial velocities as high as desired.

6. COMPARISON OF THE RESULTS WITH THE SOLUTION OF THE "NEAREST MISS" PROBLEM

The problem just solved is closely related to other optimal control problems. For example, we may consider the problem of determining the minimum distance to within which a point mass approaches the origin as a function of the initial data. Irrespective of the fact that, from the standpoint of optimal control theory, this problem is one of open-loop control, quite different from the feedback control problem considered hitherto, in mathematical terms the solution is obtained by simply replacing the

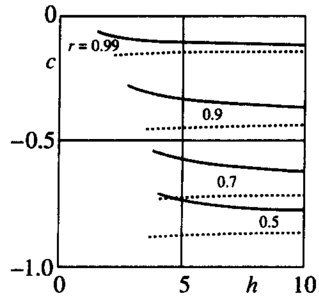

Fig. 5

dimension less variables in (5.10) in such a way that the length of the initial radius vector (not the length of the sphere's radius) equals 1, viz., $l \to 1/r$, $h \to h/\sqrt{r}$, where r is the minimum difference.

It is interesting to compare this "maximum miss" solution with the solution of the problem of minimizing the miss [10]. Figure 5 shows the curves obtained using the formulae of [10], represented by the solid curves; those obtained using (5.10) are the dotted curves. It is obvious that the comparison is feasible only for values of c and h at which the point mass bypasses the origin, that is, when the trajectory that takes the point mass exactly to the origin first bypasses it and then "returns". Accordingly, each function begins at its respective value of h. The parameter in both families is the minimum distance r. The domain above the solid curve corresponds to the case in which the minimum possible miss is greater than the corresponding value of r, and the domain beneath the dotted curve represents the case when the maximum possible deviation is less than r. It can be nearly seen (especially at fairly large values of h, e.g. close to 10) that each pair of curves of the different families, corresponding to the same value of the parameter r, forms a 'funnel" with its wider end facing the direction of small values of h. The width of the "funnel" characterizes the possibilities provided by our control. They are naturally restricted at large initial velocities, when there is not sufficient time to manoeuvre, and for r close to 1, when the spatial possibilities of manoeuvring are limited (recall that the initial distance is always 1). Thus, the two problems complement one another, enabling us to gain a better understanding of the feature of controlled motion in this system.

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